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On Bernstein and Markov-Type Inequalities for Multivariate Polynomials on Convex Bodies

András Kroó* and Szilárd Révész*

Mathematical Institute, Hungarian Academy of Sciences, Reáltona u. 13–15, Budapest H-1053, Hungary

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Let p_n be a polynomial of m variables and total degree n such that $\|p_n\|_{C(K)}=1$, where $K\subset\mathbb{R}^m$ is a convex body. In this paper we discuss some local and uniform estimates for the magnitude of grad p_n under the above conditions. © 1999 Academic Press

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INTRODUCTION

The classical inequalities of Bernstein and Markov estimating the magnitude of derivatives of univariate polynomials play a central role in approximation theory.

By the Markov Inequality for any polynomial p_n of degree at most n

$$\|p_n'\|_{C[a,b]}\leqslant\frac{2n^2}{b-a}\|p_n\|_{C[a,b]}. \tag{1}$$

The Bernstein Inequality provides the following pointwise estimate for $p_n'(x)$ when $a<x<b$

$$|p_n'(x)|\leqslant\frac{n}{\sqrt{(x-a)(b-x)}}\|p_n\|_{C[a,b]}. \tag{2}$$

Upper bounds (1) and (2) are sharp, they are attained for the Chebyshev polynomial $T_n((2x-a-b)/2(b-a))$ where $T_n(t)=\cos(n\arccos t)$ (for

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certain values of x in case of (2)). It should be noted that (1) can be deducted from (2) with the help of the following inequality of Schur

$$\|p_{n-1}\|_{C[a,b]} \leqslant \frac{2n}{b-a} \|\sqrt{(x-a)(b-x)}\, p_{n-1}(x)\|_{C[a,b]}, \tag{3}$$

where p_{n-1} is a polynomial of degree at most $n-1$.

The purpose of this paper is to study Markov and Bernstein-type inequalities for multivariate polynomials. Thus we consider the space P_n^m of polynomials

$$p_n(x) = \sum_{|k|_1 \leqslant n} a_k x^k, \qquad x \in \mathbb{R}^m, \quad a_k \in \mathbb{R}$$

of m real variables and total degree $\leqslant n$. (As usual $|k|_1 = k_1 + \cdots + k_m$ and $x^k = \prod_{j=1}^m x_j^{k_j}$, where $k = (k_1, ..., k_m) \in \mathbb{Z}_+^m$ and $x = (x_1, ..., x_m) \in \mathbb{R}^m$.) In what follows $|x|_2$ denotes the Euclidean norm of $x \in \mathbb{R}^m$, $S^{m-1} = \{x \in \mathbb{R}^m\colon |x|_2 = 1\}$ is the unit sphere in $\mathbb{R}^m$, while $B^m = \{x \in \mathbb{R}^m\colon |x|_2 \leqslant 1\}$ stands for the unit ball of $\mathbb{R}^m$. We are interested in estimating $D_y p_n(x)$, the derivative of $p_n \in P_n^m$ in the direction $y \in S^{m-1}$. In particular, this leads to estimates for the magnitude of the gradient of $p_n(x)$ given by

$$|\mathrm{grad}\ p_n(x)|_2 = \sup\{|D_y p_n(x)|\colon y \in S^{m-1}\}.$$

Naturally, in the multivariate case the results are closely related to the geometry of the underlying set $K \subset \mathbb{R}^m$ on which the uniform norm $\|p_n\|_{C(K)} = \max_{x \in K} |p_n(x)|$ of $p_n \in P_n^m$ is considered.

The first sharp Markov-type inequality in $\mathbb{R}^m$ was obtained by Kellogg [4] in 1928 in the case when $K = B^m$ is the unit ball:

$$\|\,|\mathrm{grad}\ p_n|_2\|_{C(B^m)} \leqslant n^2 \|p_n\|_{C(B^m)}, \qquad p_n \in P_n^m. \tag{4}$$

(Clearly, inequality (4) is sharp for every $n, m \in \mathbb{N}$.)

Wilhelmsen [8] gave a Markov-type estimate for an arbitrary *convex body* $K \in \mathbb{R}^m$. (A convex body in $\mathbb{R}^m$ is a convex compact set with non-empty interior.) For a convex body $K \subset \mathbb{R}^m$ denote by $w(K)$ the minimal distance between two parallel supporting hyperplanes for K. Then it is shown in [8] that

$$\|\,|\mathrm{grad}\ p_n|_2\|_{C(K)} < \frac{4n^2}{w(K)} \|p_n\|_{C(K)} \tag{5}$$

whenever $p_n \in P_n^m$ and $K \subset \mathbb{R}^m$ is a convex body. The above inequality with a different, weaker constant was given earlier by Coatmelec [3]. Note that $w(B^m) = 2$, i.e., for the unit ball the constant in (5) is twice larger than in (4). (Independently Nadzhmiddinov and Subbotin [6] verified (5) in the special case when K is a triangle in $\mathbb{R}^2$.) This leads to the interesting problem of finding the exact constant in (5). Evidently, this constant must be between 2 and 4. This question was partially resolved by Sarantopoulos [7] who found sharp Bernstein and Markov-type inequalities in the case when K is central symmetric. (Recall that K is central symmetric if and only if with proper shift it is the unit ball of some norm on $\mathbb{R}^m$.) If K is central symmetric with its center in the origin then let

$$\varphi_K(x) = \inf \left\{ \alpha > 0 \colon \frac{x}{\alpha} \in K \right\}, \qquad x \in \mathbb{R}^m \tag{6}$$

be the corresponding Minkowski functional. Then by [7], for every $p_n \in P_n^m$, $y \in S^{m-1}$ and $x \in \operatorname{Int} K$

$$|D_y p_n(x)| \leqslant \frac{n\varphi_K(y)}{\sqrt{1-\varphi_K^2(x)}} \, \|p_n\|_{C(K)}.$$

Combining this with (3) leads to the bound

$$|D_y p_n(x)| \leqslant \varphi_K(y) \, n^2 \, \|p_n\|_{C(K)}, \qquad x \in K, \quad y \in S^{m-1}.$$

Since $\varphi_K(y) \leqslant 2/w(K)$ when K is central symmetric it follows that

$$\| \, |\text{grad } p_n|_2 \, \|_{C(K)} \leqslant \frac{2n^2}{w(K)} \, \|p_n\|_{C(K)}, \tag{7}$$

$$|D_y p_n(x)| \leqslant \frac{2n}{w(K) \sqrt{1-\varphi_K^2(x)}} \, \|p_n\|_{C(K)}. \tag{8}$$

Inequalities (7) and (8) of Sarantopoulos [7] provide sharp Markov and Bernstein-type bounds for convex central symmetric sets. (Using methods of several complex variables these inequalities where established independently by Baran [1].) In a recent paper [2] by Białas-Cież and Goetgheluck it was shown that (7) fails in general for $K = \Delta_0 := \{(x, y) \in \mathbb{R}^2 \colon x, y \geqslant 0; x + y \leqslant 1\}$, but on the other hand the constant 4 in (5) can be replaced by $\sqrt{10}$ when $K = \Delta_0$. This shows that in non-symmetric case the Markov constant in (7) has to be larger than 2, but it is also possible to improve inequality (5) for some sets.

Two questions arise from the above discussion.

— How can one extend Bernstein inequality for non-symmetric convex bodies?

— How can the Markov inequality (5) be improved in non-symmetric case?

In this paper we shall address these questions. First in Section 1 we shall give some auxiliary geometric facts. Section 2 contains our main result providing a Bernstein-type estimate for derivatives of polynomials on non-symmetric convex bodies. The sharpness of this estimate will also be considered. In Section 3 we discuss possible improvements of the Markov-type inequality (5) in the non-symmetric case with triangles being studied in a more systematic way.

1. GEOMETRY

The quantity $\sqrt{(x-a)(b-x)}$ appearing in Bernstein Inequality (2) measures the distance from $x \in (a, b)$ to the endpoints of the interval (a, b). For central-symmetric convex bodies this is accomplished by the term $\sqrt{1-\varphi_K^2(x)}$ (see [7]). In order to present our Bernstein-type inequality we shall need a certain quantity $\alpha_K(x)$ introduced in [5] measuring the distance from a given $x \in \mathbb{R}^m$ to the boundary BdK of the non-symmetric convex body $K \subset \mathbb{R}^m$.

For given $a, b \in \text{Bd}\, K$ and $c \in S^{m-1}$ such that $\langle c, b-a \rangle > 0$ denote

$$S_c(a, b) := \{x \in \mathbb{R}^m\colon \langle c, a \rangle \leqslant \langle c, x \rangle \leqslant \langle c, b \rangle\},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^m$. We call $S_c(a, b)$ a *supporting strip* of K if $K \subset S_c(a, b)$. Note that the boundary of a supporting strip consists of two parallel hyperplanes supporting K at a and b. Furthermore, given $\alpha > 0$ the α-dilation of $S_c(a, b)$ is defined by

$$S_c^\alpha(a, b) := \{x \in \mathbb{R}^m\colon \langle c, a-d_\alpha \rangle \leqslant \langle c, x \rangle \leqslant \langle c, b+d_\alpha \rangle\},$$

where $d_\alpha := (\alpha-1)(b-a)/2$. For any $\alpha > 0$ and convex body $K \subset \mathbb{R}^m$ set $K_\alpha := \bigcap S_c^\alpha(a, b)$ where the intersection is taken over all supporting strips of K. Clearly, $K_1 = K$, $K_\beta \subset K_\alpha$ if $0 < \beta < \alpha$, and $K_\alpha = \alpha K$ whenever K is central-symmetric. It should be noted that when K is not central symmetric K_α does not preserve in general the “shape” of K. For instant, if K is a triangle in $\mathbb{R}^2$ and $\alpha > 1$ then K_α is a hexagon. Finally, set

$$\alpha_K(x) := \inf\{\alpha\colon x \in K_\alpha\}.$$

This quantity measuring the “distance” from $x \in \mathbb{R}^m$ to Bd K was first introduced in [5] in order to verify the relation

$$\sup_{p_n \in P_n^m} \frac{|p_n(x)|}{\|p_n\|_{C(K)}} = T_n(\alpha_K(x)), \qquad x \in \mathbb{R}^m \backslash K.$$

For central symmetric convex bodies K we have $\alpha_K(x) = \varphi_K(x)$, $x \in \mathbb{R}^m$. Also, it should be noted that $\alpha_K(x) > 1$ for $x \in \mathbb{R}^m \setminus K$, $\alpha_K(x) = 1$ on Bd K, and $\alpha_k(x) < 1$ if $x \in \text{Int } K$. Thus the closer is $\alpha_K(x)$ to 1, the closer is x to Bd K. We shall also consider another measure of distance from $x \in \text{Int } K$ to Bd K given by

$$\Delta_K(x) := \inf \frac{2\sqrt{|x-a|_2 \, |x-b|_2}}{|a-b|_2}, \qquad x \in \text{Int } K, \tag{9}$$

where the above inf is taken over all $a, b \in \text{Bd } K$ such that x belongs to the line segment connecting a and b. Clearly the above inf is attained for some $a^*, b^* \in \text{Bd } K$. Let us verify that K possesses parallel supporting hyperplanes at these points.

PROPOSITION 1. *Let $x \in \text{Int } K$ and assume that the* inf *in* (9) *is attained for $a^*, b^* \in \text{Bd } K$. Then K possesses parallel supporting hyperplanes at a^* and b^*.*

Proof. We may assume that $x = 0$ and $|b^*|_2 \leqslant |a^*|_2$. Consider the function

$$g(a) := \inf \{\alpha > 0 \colon -a/\alpha \in K\}.$$

Evidently, $g(a)$ is positive homogeneous and continuous. It is a routine exercise to show that g is convex, as well.

Furthermore whenever $a, b \in \text{Bd } K$ are such that $a = -tb$ $(t > 0)$, then $g(a) = 1/t$ and $g(b) = t$. Now minimizing the quantity

$$\frac{4\,|x-a|_2\,|x-b|_2}{|a-b|_2^2} = \frac{4\,|a|_2\,|b|_2}{|a-b|_2^2} = \frac{4t\,|b|_2^2}{((1+t)\,|b|_2)^2} = \frac{4}{2+g(a)+g(b)}$$

is equivalent to maximizing $g(a) + g(b)$. In view of $g(a)\,g(b) = 1$ the above quantity is maximal for pairs $a, b \in \text{Bd } K$ where g attains its maximal and minimal values, respectively. Thus the definition of $a^*, b^* \in \text{Bd } K$ as extremal point pair satisfying also $|b^*|_2 \leqslant |a^*|_2$ entails maximality of $g(a^*)$, i.e., $g(a) \leqslant g(a^*)$ for every $a \in K$.

Furthermore, $\text{Epi}(g) := \{(a, t) \colon a \in \mathbb{R}^m, t \in \mathbb{R}, t \geqslant g(a)\} \subseteq \mathbb{R}^{m+1}$ is closed and convex. In addition, $(a^*, g(a^*)) \in \text{Bd Epi}(g)$. By the supporting hyperplane theorem there exist $c_0 \in \mathbb{R}^m$ and $t_0 \in \mathbb{R}$ (not both of which are 0) such that

$$\langle c_0, a^* \rangle + t_0\, g(a^*) \leqslant \langle c_0, a \rangle + t_0 t$$

for every $(a, t) \in \text{Epi}(g)$.

Setting $a=a^*$ and letting $t\to\infty$ yields that $t_0\geqslant 0$. Also, if $t_0=0$ then $\langle c_0, a^*\rangle \leqslant \langle c_0, a\rangle$ for $a\in\mathbb{R}^m$, i.e., $c_0=0$, a contradiction. Thus $t_0>0$. Hence we may assume that

$$\langle c_0, a^*\rangle + g(a^*) \leqslant \langle c_0, a\rangle + t$$

for all $(a, t)\in \operatorname{Epi}(g)$. Thus for very $a\in\mathbb{R}^m$

$$\langle c_0, a^*-a\rangle \leqslant g(a)-g(a^*). \tag{10}$$

Since $g(a)\leqslant g(a^*)$, $a\in K$ it follows that

$$\langle c_0, a^*\rangle \leqslant \langle c_0, a\rangle, \qquad a\in K. \tag{11}$$

In addition, by (10)

$$\begin{aligned} g(a^*)\langle c_0, a\rangle + \langle c_0, a^*\rangle &= \langle c_0, g(a^*)\, a\rangle + \langle c_0, a^*\rangle \\ &\leqslant g(-g(a^*)\, a) - g(a^*) = g(a^*)\, g(-a) - g(a^*). \end{aligned}$$

Since $g(a^*)>0$ $(a^*\neq 0)$ we have

$$\left\langle c_0, \frac{a^*}{g(a^*)}\right\rangle \leqslant -\langle c_0, a\rangle + g(-a) - 1, \qquad a\in\mathbb{R}^m.$$

Note that $g(-a)\leqslant 1$ when $a\in K$, so we arrive at

$$\left\langle c_0, -\frac{a^*}{g(a^*)}\right\rangle \geqslant \langle c_0, a\rangle, \qquad a\in K. \tag{12}$$

Thus by (11) and (12) K possesses parallel supporting hyperplanes at a^* and $-a^*/g(a^*)=b^*$. This completes the proof of Proposition 1. ∎

Our next proposition unveils an interesting relation between quantities $\alpha_K(x)$ and $\Delta_K(x)$.

PROPOSITION 2. *For any $x\in K$ we have*

$$\Delta_K^2(x) = 1-\alpha_K^2(x). \tag{13}$$

Proof. Equation (13) is trivial when $x\in \operatorname{Bd} K$ so we may assume that $x\in \operatorname{Int} K$. Let $a, b\in \operatorname{Bd} K$ be such that x belongs to the line segment connecting a and b. We may assume that $|a-x|_2\geqslant |b-x|_2$, i.e., setting $t:=(|b-x|_2)/(|a-b|_2)$ we have that $0<t\leqslant 1/2$ and

$$\frac{|a-x|_2}{|a-b|_2}\cdot\frac{|b-x|_2}{|a-b|_2} = (1-t)\, t. \tag{14}$$

Consider a supporting strip S of K such that one of its boundary planes passes through b. Thus for some $c \in S^{m-1}$ and $\tilde{a} \in \mathrm{Bd}\, K$

$$S = S_c(\tilde{a}, b) = \{x \in \mathbb{R}^m : \langle c, \tilde{a} \rangle \leqslant \langle c, x \rangle \leqslant \langle c, b \rangle\}; \qquad S \supset K. \tag{15}$$

Let $d \in \mathbb{R}^m$, $d \neq b$, be the second point where the line through a and b intersects boundary of S. Evidently,

$$\tilde{t} := \frac{|b-x|_2}{|b-d|_2} \leqslant \frac{|b-x|_2}{|a-b|_2} = t \leqslant \frac{1}{2},$$

and therefore

$$\frac{|d-x|_2}{|b-d|_2} \cdot \frac{|b-x|_2}{|b-d|_2} = (1-\tilde{t})\, \tilde{t} \leqslant (1-t)\, t. \tag{16}$$

Consider now an arbitrary $\alpha > \alpha_K(x)$. Then $x \in K_\alpha$, i.e., $x \in S^\alpha$ for the strip S given by (15). This yields that $\tilde{t} = |b-x|_2/|b-d|_2 \geqslant (1-\alpha)/2$. Thus by (14) and (16)

$$\frac{|a-x|_2\, |b-x|_2}{|a-b|_2^2} = (1-t)\, t \geqslant (1-\tilde{t})\, \tilde{t} \geqslant \left(1 - \frac{1-\alpha}{2}\right) \frac{1-\alpha}{2} = \frac{1-\alpha^2}{4}.$$

Hence $\Delta_K^2(x) \geqslant 1 - \alpha_K^2(x)$.

Now we shall verify the converse inequality. For given $x \in \mathrm{Int}\, K$ let the inf in (9) be attained for some $a^*, b^* \in \mathrm{Bd}\, K$. Set $s^* := |x-b^*|_2/|x-a^*|_2$, $\alpha^* := (1-s^*)/(1+s^*)$, where we assume again that $|x-b^*|_2 \leqslant |x-a^*|_2$. Consider an arbitrary supporting strip $S_c(a, b)$ of K. Let r be the ray originating from a and passing through x. Denote by b_1 and b_2 the points where r exits from K and $S_c(a, b)$ respectively. We may assume, as usual, that $|x-b_2|_2 \leqslant |x-a|_2$. Furthermore, set $s := |x-b_1|_2/|x-a|_2$; $\tilde{s} := |x-b_2|_2/|x-a|_2$. Then $1 \geqslant \tilde{s} \geqslant s \geqslant s^*$, and hence

$$\frac{1-\tilde{s}}{1+\tilde{s}} \leqslant \frac{1-s}{1+s} \leqslant \frac{1-s^*}{1+s^*} = \alpha^*.$$

This yields that $1 \geqslant \tilde{s} \geqslant (1-\alpha^*)/(1+\alpha^*)$ and therefore $x \in S_c^{\alpha^*}(a, b)$ for an arbitrary supporting strip $S_c(a, b)$ of K. Thus $x \in K_{\alpha^*}$, i.e., $\alpha_K(x) \leqslant \alpha^*$. Recall that

$$\Delta_K^2(x) = \frac{4\, |x-a^*|_2\, |x-b^*|_2}{|a^*-b^*|_2^2} = \frac{4}{s^* + 2 + 1/s^*}$$
$$= 1 - (\alpha^*)^2 \leqslant 1 - \alpha_K^2(x). \quad \blacksquare$$

Now we introduce the notion of the width of the convex body K in direction $v \in S^{m-1}$. For any $x \in K$ and line $\ell_v(x)$ in direction v passing through

x denote by $d_v(x)$ the length of the segment of intersection of $\ell_v(x)$ and K. Then the quantity

$$w_v(K) := \sup_{x \in K} d_v(x)$$

will be called the width of K in direction v.

PROPOSITION 3. *For any convex body $K \subset \mathbb{R}^m$ and $v \in S^{m-1}$ we have $w_v(K) \geqslant w(K)$, where $w(K)$ is the minimal distance between two parallel supporting hyperplanes for K.*

Proof. Clearly $d_v(x)$ is a continuous function of $x \in K$, when K is convex. Thus for some $x_0 \in K$ we have $d_v(x_0) = w_v(K)$. Denote by a_0 and b_0 the points of intersection of K and the line through x_0 in the direction $v \in S^{m-1}$. Consider the set $K_0 = \operatorname{Int} K + a_0 - b_0$. We claim that $K \cap K_0 = \emptyset$. Assume that to the contrary, for some $y \in \operatorname{Int} K$ we have $y + a_0 - b_0 \in K$. Since $y \in \operatorname{Int} K$ $y - \varepsilon(a_0 - b_0) \in K$ if $\varepsilon > 0$ is small enough. But the line segment connecting points $y + a_0 - b_0,\ y - \varepsilon(a_0 - b_0) \in K$ is parallel to v and has length $(1+\varepsilon)\,|a_0 - b_0|_2 = (1+\varepsilon)\, d_v(x_0)$, a contradiction. Thus $K \cap K_0 = \emptyset$. This implies that K and K_0 can be separated by a hyperplane, i.e., with some $u \in S^{m-1}$ we have $\langle x, u\rangle \leqslant \langle y, u\rangle$ whenever $x \in K$, $y \in K_0$. Thus

$$\langle x, u\rangle \leqslant \langle y + a_0 - b_0, u\rangle, \qquad x, y \in K. \tag{17}$$

Setting $x = a_0$ in (17) yields

$$\langle b_0, u\rangle \leqslant \langle y, u\rangle, \qquad y \in K.$$

Moreover, using (17) with $y = b_0$ implies

$$\langle x, u\rangle \leqslant \langle a_0, u\rangle, \qquad x \in K.$$

Thus K possesses parallel supporting hyperplanes at a_0 and b_0, i.e., $w_v(K) = |a_0 - b_0|_2 \geqslant w(K)$. ∎

We conclude this section by some remarks and open questions concerning the quantity $\alpha_K(x)$. Evidently, $0 \leqslant \alpha_K(x) \leqslant 1$ whenever $x \in K$. Denote by

$$\alpha_K := \inf_{x \in K} \alpha_K(x)$$

the minimal value of $\alpha_K(x)$. It is not difficult to show that $\alpha_K = 0$ if and only if K is central symmetric. Thus, in particular, $\alpha_K > 0$ for every non-symmetric convex body K. To determine the size of α_K for a non-symmetric convex body K seems to be an interesting and nontrivial problem. It can be verified that $\alpha_k(x_K) \leqslant (m-1)/(m+1)$ if x_K is the *center of mass* of the convex body $K \subset \mathbb{R}^m$. Thus, in particular, $\alpha_k \leqslant (m-1)/(m+1)$ for every $K \subset \mathbb{R}^m$. Moreover, if K is any finite convex body arising from cutting a cone by a hyperplane in $\mathbb{R}^m$, then $\alpha_K = (m-1)/(m+1)$ with x_K being the only point in K with $\alpha_K(x_K) = (m-1)/(m+1)$. We conjecture that for every convex body K in $\mathbb{R}^m$ there exists a *unique* point $x \in K$ such that $\alpha_k(x) = \alpha_K$.

2. A BERNSTEIN-TYPE INEQUALITY FOR MULTIVARIATE POLYNOMIALS ON NON-SYMMETRIC CONVEX BODIES IN $\mathbb{R}^M$

In this section we shall apply the geometric results of the previous section in order to derive a Bernstein-type inequality for non-symmetric convex bodies. One approach to this problem consists in utilizing the technique of Wilhelmsen [8], who essentially linearized this problem by passing to proper lines. Let us outline this approach.

Consider an arbitrary $y \in S^{m-1}$ and $x \in \operatorname{Int} K$. Let $S_y(a, b)$ be a supporting strip of K, where $a, b \in \operatorname{Bd} K$. Let c be the point where the ray originating from a and passing through x exits from K. We may assume without loss of generality that $\operatorname{dist}(x, H_a) \geqslant \frac{1}{2}\operatorname{dist}(H_a, H_b) \geqslant w(K)/2$, where H_a and H_b are the boundary hyperplanes of $S_y(a, b)$ passing through a and b, respectively. Set $u := (a-c)/|a-c|_2 \in S^{m-1}$. Then using the classical Bernstein Inequality (2) on the line segment connecting points a and c we have when $\|p_n\|_{C(K)} = 1$

$$|D_u p_n(x)| \leqslant \frac{n}{\sqrt{|x-a|_2 \, |x-c|_2}} \leqslant \frac{2n}{|a-c|_2 \, \Delta_K(x)}. \tag{18}$$

We can choose $y \in S^{m-1}$ so that it is the unit vector in direction of $\operatorname{grad} p_n(x)$. Clearly, $|D_u p_n(x)| = |\operatorname{grad} p_n(x)|_2 \cos\varphi$, where φ is the angle between u and y. Moreover, $|a-c|_2 \cos\varphi = \operatorname{dist}(c, H_a) \geqslant \operatorname{dist}(x, H_a) \geqslant w(K)/2$. Thus we have by (18)

$$|\operatorname{grad} p_n(x)|_2 = \frac{|D_u p_n(x)|}{\cos\varphi} \leqslant \frac{4n}{w(K)\, \Delta_K(x)}.$$

Furthermore, (13) yields that

$$|\operatorname{grad} p_n(x)|_2 \leqslant \frac{4n}{w(K) \sqrt{1-\alpha_K^2(x)}}, \tag{19}$$

where $\|p_n\|_{C(K)} = 1$.

The quantity $\sqrt{1-\alpha_K^2(x)}$ appearing in (19) can be considered as the "Bernstein-factor" corresponding to $\sqrt{1-\varphi_K^2(x)}$ of (8) which holds in the central symmetric case. Thus in non-symmetric case the constant in (19) is twice larger than the one in (8). (This is analogous to the increase of constant in the Markov-type inequality (5) relative to (7) which holds in central symmetric case.)

We shall present below another approach to Bernstein inequality on non-symmetric convex bodies which will enable us to replace the constant 4 in (19) by $2\sqrt{2}$. This approach is based on a more delicate technique when the problem is linearized by inscribing *ellipses* into K (and not line segments). We shall also show that for every $K \subset \mathbb{R}^m$ the optimal constant in (19) can not be smaller than 2, in general.

THEOREM 1. *Let $K \subset \mathbb{R}^m$ be a convex body, $p_n \in P_n^m$, $\|p_n\|_{C(K)} = 1$; $n, m \in \mathbb{N}$. Then for every $y \in S^{m-1}$ and $x \in \operatorname{Int} K$ we have*

$$|D_y p_n(x)| \leqslant \frac{2n}{w_y(K)\sqrt{1-\alpha_K(x)}} \leqslant \frac{2\sqrt{2}\,n}{w_y(K)\sqrt{1-\alpha_K^2(x)}}. \tag{20}$$

Recall that by Proposition 3 $w_y(K) \geqslant w(K)$ for every $y \in S^{m-1}$. Thus Theorem 1 yields the next

COROLLARY 1. *For every convex body $K \subset \mathbb{R}^m$ and polynomial $p_n \in P_n^m$*

$$\left\|\sqrt{1-\alpha_K^2(x)}\,|\operatorname{grad} p_n(x)|_2\right\|_{C(K)} \leqslant \frac{2\sqrt{2}\,n}{w(K)}\|p_n\|_{C(K)}. \tag{21}$$

Our next result shows that the constant in the first inequality in (20) is optimal in general, while the constant of (21) might differ from the best possible by at most $\sqrt{2}$.

THEOREM 2. *Let $K \subset \mathbb{R}^m$ be a convex body, and assume that n is sufficiently large, so that $\cos(\pi/2n) \geqslant \alpha_K$. Then for every $x \in \operatorname{Int} K$ such that $\sin n \arccos \alpha_k(x) = 1$ there exist a $v \in S^{m-1}$ and $p_n \in P_n^m$, $\|p_n\|_{C(K)} = 1$, so that*

$$D_v p_n(x) = \frac{2n}{w_v(K)\sqrt{1-\alpha_K^2(x)}}. \tag{22}$$

Note that when $x \in \operatorname{Int} K$ the quantity $\alpha_K(x)$ attains all values between α_K and 1. Thus if $\cos(\pi/2n) \geqslant \alpha_K$ the set of those points $x \in K$ for which $\sin(n \arccos \alpha_K(x)) = 1$ is nonempty. On the other hand $\alpha_K \leqslant (m-1)/(m+1)$ implies the existence of such points x for $n \geqslant n_0(m) = \pi/(2\arccos((m-1)/(m+1)))$ independently of the body K.

EXAMPLE 1. Let $K=\Delta_0=\{(x, y)\in\mathbb{R}^2: x, y\geqslant 0; x+y\leqslant 1\}$ $(m=2)$. Then it is easy to show that $w(K)=\sqrt{2}/2$ and for every $(x, y)\in \text{Int}\,\Delta_0$, $1-\alpha_K(x, y)=2\min\{1-x-y, x, y\}$. Thus by the first inequality in (20) for every $p_n\in P_n^m$

$$|\text{grad}\, p_n(x, y)|_2\leqslant\frac{\sqrt{2}\,n}{\sqrt{\min\{1-x-y, x, y\}}}\|p_n\|_{C(\Delta_0)}.$$

Proof of Theorem 1. Let $G, H\in \text{Bd}\,K$ be such that $|G-H|_2=w_y(K)$, and the line through G and H is parallel to y. Consider also $F, E\in \text{Bd}\,K$ such that the line through F and E is parallel to y and the point $x\in \text{Int}\,K$ belongs to this line. Now we shall reduce our considerations to the 2-dimensional plane spanned by G, H, F, E. (The case when x belongs to the line segment connecting G and H is trivial.) By a suitable choice of the coordinate axises in this plane we may assume that $y=(0, 1)$ and $G=(-a, g)$, $H=(-a, h)$, $E=(a, e)$, $F=(a, f)$ where $a>0$, $g>h$ and $f>e$. Let B be the point where the diagonals of the trapezoid $GFEH$ intersect, and denote by C the intersection of the line through x and B with the segment connecting H and G. We may assume without loss of generality that the midpoint of the segment connecting C and x coincides with the origin, i.e., $x=-C$. Now set

$$\lambda:=\frac{|E-F|_2}{|G-H|_2}=\frac{f-e}{g-h}, \qquad 0<\lambda\leqslant 1. \tag{23}$$

Let $\mu\in(-1, 1)$ be such that $x=(a, ((1+\mu)/2)\,f+((1-\mu)/2)\,e)$. Then routine similarity arguments yield that $C=(-a, ((1+\mu)/2)\,h+((1-\mu)/2)\,g)$. Consider the ellipse

$$\tilde{x}(t):=x\cos t+by\sin t, \qquad 0\leqslant t\leqslant 2\pi, \tag{24}$$

where

$$b:=\tfrac{1}{2}|G-H|_2\sqrt{\lambda(1-\mu^2)}=\tfrac{1}{2}w_y(K)\sqrt{\lambda(1-\mu^2)}. \tag{25}$$

We shall verify now that this ellipse is inscribed into the trapezoid $GHEF$. Since $\tilde{x}(0)=x$, $\tilde{x}'(0)=by$, $\tilde{x}(\pi)=-x=C$ and $\tilde{x}'(\pi)=-by$ it follows that $\tilde{x}(t)$ passes through x and C and has vertical tangent at these points. Thus it remains to verify that the ellipse $\tilde{x}(t)$ is enclosed between segments GF and HE. Clearly, $\tilde{x}(t)$ is below GF provided that

$$\langle\tilde{x}(t), u\rangle\leqslant\langle F, u\rangle, \tag{26}$$

where $u:=(g-f, 2a)$. We have

$$\langle F, u\rangle=a(g-f)+2af=a(g+f).$$

Moreover, by (24)

$$\langle \tilde{x}(t), u\rangle^2 \leqslant a^2(4b^2 + (g + \mu f + (1-\mu)\, e)^2).$$

Thus (26) will hold provided that

$$\begin{aligned} 4b^2 &\leqslant (g+f)^2 - (g+\mu f + (1-\mu)\, e)^2 \\ &= 2(1-\mu)(f-e)\left(g + \frac{1+\mu}{2} f + \frac{1-\mu}{2} e\right). \end{aligned} \tag{27}$$

Since $x = (a, ((1+\mu)/2)\, f + ((1-\mu)/2)\, e) = -C = (a, -((1+\mu)/2)\, h - ((1-\mu)/2)\, g)$ it follows that

$$\frac{1+\mu}{2} f + \frac{1-\mu}{2} e = -\frac{1+\mu}{2} h - \frac{1-\mu}{2} g.$$

Using this we can write (27) as

$$4b^2 \leqslant 2(1-\mu)(f-e) \cdot \frac{1+\mu}{2} (g-h) = (1-\mu^2)(f-e)(g-h). \tag{28}$$

But by (25) and (23)

$$4b^2 = (g-h)^2\, \lambda(1-\mu^2) = (g-h)(f-e)(1-\mu^2).$$

Thus (28) holds which in turn yields that (27) and (26) are true, as well. Hence $\tilde{x}(t)$ is below GF. Analogously it can be shown that $\tilde{x}(t)$ lies above segment HE. Finally, this leads to the conclusion that the ellipse $\tilde{x}(t)$, $0 \leqslant t \leqslant 2\pi$, is inscribed into the trapezoid $GHEF$, i.e., into K, as well. Consider now the trigonometric polynomial $t_n(t) = p_n(\tilde{x}(t))$ of degree at most n $(0 \leqslant t \leqslant 2\pi)$. Since ellipse $\tilde{x}(t)$ is inscribed into K and $\|p_n\|_{C(K)} = 1$ it follows that $\|t_n\|_{C[0, 2\pi]} \leqslant 1$. Then by the Bernstein Inequality for trigonometric polynomials $|t_n'(0)| \leqslant n$. Using that $t_n'(0) = bD_y p_n(x)$ be obtain by (25)

$$|D_y p_n(x)| \leqslant \frac{n}{b} = \frac{2n}{w_y(K) \sqrt{\lambda(1-\mu^2)}}. \tag{29}$$

Let us first consider the case when $\lambda = 1$. Then by Proposition 2

$$\lambda(1-\mu^2) = 1-\mu^2 = \frac{4\, |F-x|_2\, |E-x|_2}{|F-E|_2^2} \geqslant \Delta_K^2(x) = 1 - \alpha_K^2(x).$$

This together with (29) yields an estimate which is even stronger than (20).

It remains to consider the case when $0<\lambda<1$, i.e., $|E-F|_2<|G-H|_2$. Then the line through H and x must intersect the line through G and F at some point Q. Furthermore, the line through H and x intersects Bd K at a point R located on the segment connecting x and Q. Then

$$\frac{(1-\mu)\,\lambda}{2}=\frac{|F-x|_2}{|G-H|_2}=\frac{|Q-x|_2}{|Q-H|_2}\geqslant\frac{|R-x|_2}{|R-H|_2}:=t. \tag{30}$$

Furthermore, by Proposition 2

$$1-\alpha_K^2(x)=\Delta_K^2(x)\leqslant 4t(1-t).$$

This easily yields that $t\geqslant(1-\alpha_K(x))/2$. Hence by (30) $(1-\mu)\,\lambda\geqslant 1-\alpha_k(x)$. Similarly it can be shown that $(1+\mu)\,\lambda\geqslant 1-\alpha_k(x)$ where $\mu\in(-1,1)$. Therefore $(1-\mu^2)\,\lambda\geqslant 1-\alpha_K(x)$, and applying this in (29) yields (20). This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Let $a^*, b^*\in$ Bd K be such that

$$\Delta_K^2(x)=4\,|x-a^*|_2\,|x-b^*|_2/|a^*-b^*|_2^2$$

and $x=ta^*+(1-t)\,b^*$ with some $0<t\leqslant 1/2$. By Proposition 1 K possesses parallel supporting hyperplanes at a^* and b^*, i.e., with some $c\in S^{m-1}$

$$\langle c, a^*\rangle\leqslant\langle c, y\rangle\leqslant\langle c, b^*\rangle,\qquad y\in K. \tag{31}$$

In particular, this also implies that $w_v(K)=|a^*-b^*|_2$, where $v:=(b^*-a^*)/|a^*-b^*|_2$. Hence

$$\Delta_K(x)=\frac{2}{w_v(K)}\sqrt{|x-a^*|_2\,|x-b^*|_2}=2\sqrt{t(1-t)}. \tag{32}$$

Set now $\beta:=2/w_v(K)\langle c, v\rangle$, and

$$p_n(y):=T_n\left(\beta\left\langle c, y-\frac{a^*+b^*}{2}\right\rangle\right),$$

where $T_n(x)=\cos n\arccos x$ is the Chebyshev polynomial.

Note that by (31), $\langle c, v\rangle>0$, i.e., $\beta>0$. Using (31) we obtain that

$$-1\leqslant\beta\left\langle c, y-\frac{a^*+b^*}{2}\right\rangle\leqslant 1,\qquad y\in K,$$

i.e., $\|p_n\|_{C(K)} \leqslant 1$. Furthermore, set

$$\gamma(x) := \beta \left\langle c, x - \frac{a^* + b^*}{2} \right\rangle = \beta \left\langle c, \left(\frac{1}{2} - t\right)(b^* - a^*) \right\rangle$$
$$= \left(\frac{1}{2} - t\right) \beta w_v(K) \langle c, v \rangle = 1 - 2t. \tag{33}$$

We have then

$$D_v p_n(x) = \langle \text{grad } p_n(x), v \rangle = \beta T'_n(\gamma(x)) \langle c, v \rangle$$
$$= \frac{2n}{w_v(K)} \cdot \frac{\sin(n \arccos \gamma(x))}{\sqrt{1 - \gamma^2(x)}}. \tag{34}$$

Note that by (32) and (33) $\gamma(x) = \sqrt{1 - \Delta_K^2(x)} = \alpha_K(x)$. Thus $\sin(n \arccos \gamma(x)) = \sin(n \arccos \alpha_K(x)) = 1$ and by (34)

$$D_v p_n(x) = \frac{2n}{w_v(K)\, \Delta_K(x)} = \frac{2n}{w_v(K) \sqrt{1 - \alpha_K^2(x)}}. \quad \blacksquare$$

3. SOME IMPROVEMENTS OF THE MARKOV-TYPE INEQUALITY FOR MULTIVARIATE POLYNOMIALS ON NON-SYMMETRIC CONVEX BODIES IN $\mathbb{R}^m$

In this last section we shall discuss possible improvements of the Markov-type inequality (5) of Wilhelmsen. In the previous section it was shown how the technique of "inscribed ellipses" improves the constant in Bernstein-type inequalities on non-symmetric bodies. (Inequality (20) is substantially sharper than estimate (19) derived using Wilhelmsen's technique.) One would except that this method should yield a similar improvement of the Markov-type inequality. Unfortunately this approach gives a rather modest decrease in constant in (5). Namely we shall verify the following

THEOREM 3. *Let $K \subset \mathbb{R}^m$ be a convex body. Then for every $p_n \in P_n^m$*

$$\| |\text{grad } p_n|_2 \|_{C(K)} \leqslant \frac{4n^2 - 2n}{w(K)} \|p_n\|_{C(K)}. \tag{35}$$

Proof. We may assume that $\|p_n\|_{C(K)} = 1$. For an arbitrary $y \in S^{m-1}$ let $G, H \in \text{Bd } K$ be such that $|G - H|_2 = w_y(K)$, and the line through G and H is parallel to y. Set $M = (G + H)/2$. Furthermore, choose an arbitrary $x \in K$, and set $x(t) = (1 - t)\, x + tM$, $0 \leqslant t \leqslant 1$. We shall reduce our considerations now to the triangle Δ with vertices x, G and H. Our goal is to estimate

$D_y p_n$ at the point $x(t)$ lying inside this triangle. Repeating the procedure used in the proof of Theorem 1 (with K replaced by Δ) we shall arrive again at inequality (29) with $\lambda = t$ and $\mu = 0$, i.e.,

$$|D_y p_n(x(t))| \leqslant \frac{2n}{w_y(\Delta)\sqrt{t}} = \frac{2n}{w_y(K)\sqrt{t}}, \qquad 0 < t < 1. \tag{36}$$

Set $g(t) := D_y p_n(x(1-t^2))$, $-1 < t < 1$. Evidently, g is a univariate algebraic polynomial of degree at most $2n-2$. Moreover by (36)

$$|g(t)| \leqslant \frac{2n}{w_y(K)\sqrt{1-t^2}}, \qquad -1 < t < 1.$$

Then by the Schur Inequality (3)

$$\|g\|_{C[-1,1]} \leqslant (2n-1) \cdot \frac{2n}{w_y(K)} = \frac{4n^2 - 2n}{w_y(K)}.$$

Thus

$$|D_y p_n(x)| = |g(1)| \leqslant \frac{4n^2 - 2n}{w_y(K)}.$$

Since $x \in K$ and $y \in S^{m-1}$ where chosen arbitrarily this yields the statement of Theorem 3. ∎

We shall achieve a more significantly improvement of the Markov constant in case when K is a triangle in $\mathbb{R}^2$. As it was mentioned in the Introduction the Markov constant 4 in Wilhelmsen's inequality (5) was replaced in [2] by $\sqrt{10}$ in the special case when $K = \Delta_0 = \{(x, y) \in \mathbb{R}^2 : x, y \geqslant 0;\ x + y \leqslant 1\}$. We shall replace below the constant 4 in (5) by a smaller quantity for *arbitrary* triangles in $\mathbb{R}^2$.

Let us consider a triangle Δ in $\mathbb{R}^2$ with sides a, b, c and corresponding angles α, β, γ. We may assume that $c \leqslant b \leqslant a$, i.e., $\gamma \leqslant \beta \leqslant \alpha$. With this assumption set

$$M(\Delta) := \frac{2}{a}\sqrt{a^2 + b^2 + 2ab\cos\gamma}.$$

Clearly, $M(\Delta) \leqslant 2\sqrt{2 + 2\cos\gamma} < 4$.

On the other hand, elementary geometric arguments yield that whenever $0 < \gamma \leqslant \beta \leqslant \alpha \leqslant \pi/2$ we have

$$M(\Delta) \geqslant \sqrt{10}. \tag{37}$$

Moreover, equality in (37) is obtained only if $\Delta \cong \Delta_0$.

THEOREM 4. *Assume that* $\Delta \subset \mathbb{R}^2$ *is a triangle with angles* $0 < \gamma \leqslant \beta \leqslant \alpha \leqslant \pi/2$. *Then for every* $p_n \in P_n^2$ *we have*

$$\| |\text{grad } p_n|_2 \|_{C(\Delta)} \leqslant \frac{M(\Delta)\, n^2}{w(\Delta)} \|p_n\|_{C(\Delta)}. \tag{38}$$

Since $M(\Delta) < 4$ this estimate improves (5) for every acute and right triangle. When $\Delta = \Delta_0$ we have $M(\Delta_0) = \sqrt{10}$ and thus the Markov inequality given for Δ_0 in [2] is recovered. Note also that for equilateral triangle with side h we have $M(\Delta) = \sqrt{12}$ and $w(\Delta) = \sqrt{3}\, h/2$. Hence we obtain from (38) in this case

$$\| |\text{grad } p_n|_2 \|_{C(\Delta)} \leqslant \frac{4n^2}{h} \|p_n\|_{C(\Delta)}.$$

Theorem 4 can be applied to obtain an improvement of estimate (5) for *every* triangle. It yields the next

COROLLARY 2. *Let* Δ *be an arbitrary triangle in* $\mathbb{R}^2$ *with* $\gamma > 0$ *being its smallest angle. Then for every* $p_n \in P_n^2$ *we have*

$$\| |\text{grad } p_n|_2 \|_{C(\Delta)} \leqslant \frac{4 \cos(\gamma/2)}{w(\Delta)} n^2 \|p_n\|_{C(\Delta)}.$$

Proof of Theorem 4. Let Δ have vertices $C = (0, 0)$, $B = (a, 0)$, $A = (b \cos \gamma, b \sin \gamma)$. Set $\Delta_1 = (1/3)\, \Delta$; $\Delta_2 = \Delta_1 + (2/3)\, B$; $\Delta_3 = \Delta_1 + (2/3)\, A$. Then $\Delta = \Omega \cup (\bigcup_{j=1}^3 \Delta_j)$, where Ω is a central symmetric hexagon with $w(\Omega) = (2/3)\, w(\Delta)$. Let $p_n \in P_n^2$ be such that $\|p_n\|_{C(\Delta)} = 1$. Then applying Sarantopoulos' inequality (7) on Ω yields

$$\| |\text{grad } p_n|_2 \|_{C(\Omega)} \leqslant \frac{2n^2}{w(\Omega)} = \frac{3n^2}{w(\Delta)}. \tag{39}$$

It remains now to estimate $|\text{grad } p_n|_2$ on Δ_1 (Δ_2 and Δ_3 are isometric to Δ_1). We shall use Wilhelmsen's method on Δ_1. For $x \in \Delta_1$ and $y \in \mathbb{R}^2$ denote by $\ell(x, y)$ the segment of intersection of Δ with the line trough x in the direction y. Then by the univariate Markov Inequality $|D_y p_n(x)| \leqslant 2n^2/|\ell(x, y)|_2$. Denoting by φ the angle between y and grad $p_n(x)$ we obtain

$$|\text{grad } p_n(x)|_2 \leqslant \frac{2n^2}{|\ell(x, y)|_2 \cos \varphi} \qquad \left(0 \leqslant \varphi < \frac{\pi}{2}\right). \tag{40}$$

Set $v := (2/3)\, B = ((2/3)\, a, 0)$, $y := B - x$, $u := \text{grad } p_n(x)/|\text{grad } p_n(x)|_2$, and denote by ψ the angle between v and u.

Case 2. $\gamma-\pi/2\leqslant\psi\leqslant-\beta+\pi/2$. In this case $\langle u, v\rangle\geqslant 0$, $\langle u, y\rangle\geqslant 0$, and $\langle u, y-v\rangle\geqslant 0$. Thus

$$|\langle y, u\rangle|\geqslant|\langle v, u\rangle|=\tfrac{2}{3}\,a\cos\psi\geqslant\tfrac{2}{3}\,a\min\{\sin\beta, \sin\gamma\}\geqslant\tfrac{2}{3}\,w(\Delta).$$

Then $|\ell(x, y)|_2\cos\varphi\geqslant|\langle y, u\rangle|\geqslant 2w(\Delta)/3$ and applying this with (40) yields again

$$|\text{grad }p_n(x)|_2\leqslant\frac{3n^2}{w(\Delta)}.$$

Case 2. $-\beta+\pi/2\leqslant\psi\leqslant\pi/2$. This case can be treated similarly to Case 1 if we replace ψ by $\psi' := \gamma-\psi$, v by $v' := (2/3)\,A$ and y by $y' := A-x$.

Case 3. $-\pi/2\leqslant\psi\leqslant\gamma-\pi/2$. Let L be the line $\{tu\colon t\in\mathbb{R}\}$, where as above $u\in S^1$ is the direction of grad $p_n(x)$. Denote by A_1 and B_1 the projections to L of vertices A and B, respectively. Since $-\pi/2\leqslant\psi\leqslant\gamma-\pi/2$ it is obvious that $C=(0, 0)$ is between A_1 and B_1 (on the line L). Moreover

$$|\ell(x, B-x)|_2\cos\varphi_B\geqslant|B_1|_2,$$

$$|\ell(x, A-x)|_2\cos\varphi_A\geqslant|A_1|_2,$$

where φ_A and φ_B are the angles between u and $B-x$ and $A-x$, respectively.

Thus using (40) we obtain

$$|\text{grad }p_n(x)|_2\leqslant\frac{2n^2}{\mu}, \tag{41}$$

where

$$\mu := \min_{\substack{u\in S^1\\ -\pi/2\leqslant\psi\leqslant\gamma-(\pi/2)}}\max\{|A_1|_2, |B_1|_2\}.$$

Clearly, the above minimum is obtained when $|A_1|_2=|B_1|_2$, i.e., $a\cos\psi=b\,|\cos(\gamma-\psi)|$. Solving the last equation for ψ yields

$$\cos\psi=\frac{b\sin\gamma}{\sqrt{a^2+b^2+2ab\cos\gamma}}.$$

Thus we have in this case

$$\mu=|A_1|_2=|B_1|_2=a\cos\psi=\frac{ab\sin\gamma}{\sqrt{a^2+b^2+2ab\cos\gamma}}=\frac{2w(\Delta)}{M(\Delta)}.$$

Hence by (41)

$$|\text{grad}\ p_n(x)|_2 \leqslant \frac{M(\Delta)\, n^2}{w(\Delta)}, \qquad x \in \Delta_1,$$

which is the same as (38). Recall that in all other cases we obtained estimate (39), which is better than (38) in view of the fact that $M(\Delta) \geqslant \sqrt{10} > 3$. This completes the proof of Theorem 4. ∎

Proof of Corollary 2. For acute triangles the Corollary follows from the estimate $M(\Delta) \leqslant 2\sqrt{2+2\cos\gamma}$ combined with (38).

Assume now that Δ is an obtuse triangle with angles $\gamma \leqslant \beta < \alpha$ ($\alpha > \pi/2$) and corresponding sides $c \leqslant b \leqslant a$. Drawing perpendicular lines to a and b, at the point A, Δ can be covered by two right triangles Δ_1 and Δ_2 with angles γ, $(\pi/2)-\gamma$, $\pi/2$ and β, $(\pi/2)-\beta$, $\pi/2$, respectively. Evidently, $w(\Delta_1) = w(\Delta_2) = w(\Delta)$. Furthermore, noting that $\gamma \leqslant (\pi/2) - \gamma < \pi/2$ we have

$$\begin{aligned} M(\Delta_1) &= \frac{2\cos\gamma}{b}\sqrt{\frac{b^2}{\cos^2\gamma} + b^2 + 2b^2} \\ &= 2\sqrt{1+3\cos^2\gamma} \leqslant 2\sqrt{2+2\cos\gamma}. \end{aligned} \tag{42}$$

For the triangle Δ_2 two cases may occur.

Case 1. $\beta \leqslant \pi/4$. Then $\beta \leqslant (\pi/)2 - \beta < \pi/2$ and

$$\begin{aligned} M(\Delta_2) &= \frac{2\cos\beta}{c}\sqrt{\frac{c^2}{\cos^2\beta} + c^2 + 2c^2} \\ &= 2\sqrt{1+3\cos^2\beta} \leqslant 2\sqrt{1+3\cos^2\gamma} \leqslant 2\sqrt{2+2\cos\gamma}. \end{aligned} \tag{43}$$

Case 2. $\beta > \pi/4$, i.e., $(\pi/2) - \beta < \beta < \pi/2$. Then

$$\begin{aligned} M(\Delta_2) &= \frac{2\cos\beta}{2}\sqrt{\frac{c^2}{\cos^2\beta} + c^2\tan^2\beta + 2c^2\tan^2\beta} \\ &= 2\sqrt{1+3\sin^2\beta} = 2\sqrt{1+3\cos^2\left(\frac{\pi}{2}-\beta\right)} \\ &\leqslant 2\sqrt{1+3\cos^2\gamma} \leqslant 2\sqrt{2+2\cos\gamma}. \end{aligned} \tag{44}$$

Thus by (42)–(44), $M(\Delta_1)$, $M(\Delta_2) \leqslant 4\cos(\gamma/2)$. Since $w(\Delta_1) = w(\Delta_2) = w(\Delta)$ we obtain from Theorem 4 (applied to Δ_1 and Δ_2) the needed estimate for obtuse triangles, as well. ∎

We conclude this paper by a remark concerning polynomials on any real Banach space (see [7] for the corresponding definition). The main results of this paper can be verified in the same manner for these polynomials.

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